

Generalized Gauss Laguerre Quadrature in the Lévy Base Correlation Algorithm

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Abstract

In an earlier paper we introduced Lévy base correlation. In this paper we outline the Gauss Laguerre integration in the Lévy base correlation algorithm. The disadvantage is obvious: the quadrature formula depends on the base correlation parameter and consequently there is computational overhead in generating the quadrature formula. However accuracy considerations clearly rule out classical tabulated Gauss Laguerre quadrature. The algorithms presented here are efficient in computing the Generalized Gauss Laguerre quadrature formulas, making the Lévy Base Correlation model a very tractable algorithm.

1 Introduction

In an earlier paper we introduced Lévy base correlation. In this paper we describe the Gauss Laguerre integration in the Lévy base correlation algorithm. The paper is organized as follows. In section 2 we review the generic 1-factor model for valuation of CDO tranches and define Lévy base correlation. The need for Generalized Gauss Laguerre quadrature is outlined in section 3. In section 4 we present two algorithms to compute the abscissas and weights of the quadrature formulas. Finally conclusions are presented in section 5.

2 Generic One Factor Model

The 1-factor Gaussian copula model using the so called *recursion algorithm* was first introduced by Andersen et al. [ASB03] and is in widespread use by market participants. In what follows we give a brief description of the generic 1-factor algorithm.

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Consider a portfolio of N firms and fix a time horizon T . It is standard market practice to assume the default process to follow an inhomogeneous Poisson process and as such for any $0 \leq t \leq T$ the default times τ_i and default intensities $\lambda_i(t)$, $i = 1, \dots, N$, satisfy

$$\mathbb{P}(\tau_i > t) = \exp\left(-\int_0^t \lambda_i(u) du\right) \quad (1)$$

where \mathbb{P} is the risk-neutral probability measure. In a 1-factor model of portfolio defaults, a single systemic factor X is introduced, conditional upon which all default probabilities are independent. The single name survival probabilities $\mathbb{P}(\tau_i > t)$ are typically implied from the credit default swap (CDS) market.

The key step in valuing Collateralized Debt Obligation (CDO) tranches is to compute the joint loss distribution. In the recursion algorithm one computes a discretized version of the conditional loss distribution by means of a simple recursion formula. A loss unit u is chosen so that within a certain tolerance, losses can be represented by integers. For the iTraxx Europe Main portfolio with an assumed uniform recovery rate of 40%, the loss unit is 0.48%. We denote by $P^{(i)}(l, t|X)$ the probability of l losses (in terms of the loss unit u) at time t with i names conditional on the market factor X . Recalling that conditional on X all default probabilities are independent, we can write that $P^{(i)}(l, t|X)$ is the sum of two terms:

$$P^{(i)}(l, t|X) = P^{(i-1)}(l, t|X)\mathbb{P}(\tau_i > t|X) + P^{(i-1)}(l - \omega_i, t|X)(1 - \mathbb{P}(\tau_i > t|X)), \quad (2)$$

where ω_i is the number of loss units incurred by a default of the i th name. The unconditional loss distribution is found by integrating over the market factor

$$P(l, t) = \int_{\Omega_X} P(l, t|X) f(X) dX, \quad (3)$$

where $f(X)$ is the density of the probability distribution of the market factor X . In the base correlation framework, the expected loss on a tranche is computed as the difference of the expected loss of two equity tranches

$$\mathbb{E}L[A-D] = \mathbb{E}L[0-D; \rho_D] - \mathbb{E}L[0-A; \rho_A]. \quad (4)$$

In the so-called *latent variable* model default occurs when a certain (latent) variable A_i (usually the return) falls below a certain threshold K_i that is implied from CDS quotes. The *market* or *systemic* factor X and the *idiosyncratic* factor $X^{(i)}$ are random variables whose functional form depends on model assumptions. In the generic one factor Lévy model the latent variable is represented as

$$A_i = X_\rho + X_{1-\rho}^{(i)}, \quad i = 1, \dots, N, \quad (5)$$

where X and $X^{(i)}$ are independent and identically distributed variates and each A_i has the same infinitely divisible distribution function H_1 . A distribution is said to be *infinitely divisible*, if for every positive integer n , the characteristic function $\phi(u) = \mathbb{E}[iuX]$ is also the n th power of a characteristic function. Given an infinitely divisible distribution, a stochastic process,

$X = \{X_t, t \geq 0\}$, can be constructed. This so-called Lévy process starts at zero $X_0 = 0$, has independent and stationary increments and the distribution of the increments $X_{t+s} - X_s$, has $(\phi(u))^t$ as characteristic function. The cumulative distribution function of X_t and its inverse are denoted by H_t and $H_t^{[-1]}$ respectively. We normalize the distribution so that $\mathbb{E}[X_1] = 0$ and $\text{Var}[X_1] = 1$. Hence one has $\text{Var}[X_t] = t$. Note that for $i \neq j$, we have $\text{Corr}[A_i, A_j] = \rho$. The threshold implied from the CDS risk neutral probability of default $p_i(t)$ is given by

$$K_i(t) = H_1^{[-1]}(p_i(t)). \quad (6)$$

The conditional default probability of firm i given the value x for the systemic factor is given by

$$p_i(x; t) = H_{1-\rho}(K_i(t) - x). \quad (7)$$

We consider two choices for the distributions of the latent variables. First, note that the classical Gaussian copula model is a special case of this generic one factor model, in which the normal distribution is used. Second, we use a shifted gamma distribution and set $X_t = \sqrt{at} - G_t$, in which G_t follows a $\text{gamma}(at, \sqrt{a})$ distribution so that $\mathbb{E}[X_1] = 0$ and $\text{Var}[X_1] = 1$. Both the cumulative distribution function $H_t(x; a)$ of X_t , and its inverse $H_t^{[-1]}(x; a)$, can easily be obtained from the gamma cumulative distribution function and its inverse. Lévy base correlation is defined as the base correlation in the shifted gamma model with fixed $a = 1$. Hence the Exponential(1) distribution is used for each A_i .

For more details on the model we refer to Albrecher et al. [ALS06], where generic one factor Lévy models are outlined, our earlier paper, Garcia et al. [GGMS07], in which we have introduced Lévy base correlation, and to our historical study, Garcia and Goossens [GG07], where we compare the Gaussian and Lévy base correlation models.

Several authors have described one factor models using distributions other than the standard normal distribution, see e.g. Baxter [Bax06], Hooda [Hoo06], Guégan and Houdain [GH05], Kalemanova et al. [KSW07] and Moosbrucker [Moo06]. For more details on Lévy processes we refer to Bertoin [Ber96], Sato [Sat00] and Kyprianou [Kyp06], and to Schoutens [Sch03] for applications of Lévy processes in finance.

3 Generalized Gauss Laguerre quadrature

We now turn to the evaluation of the integral in (3) to compute the unconditional loss distribution in case the (shifted) gamma distribution is used. The probability density function of the gamma distribution can be expressed as follows

$$f(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) \quad (8)$$

for $x > 0$, where $\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) dt$ denotes the gamma function. Note that we are using the parametrization of the gamma distribution with shape parameter $a > 0$ and rate parameter $b > 0$. An alternative parametrization is to use the scale parameter $\theta = b^{-1}$. The support of the

shifted gamma distribution is $-\infty < x < \gamma$, where $\gamma = \sqrt{a}$. Computing the expected value of a function $f(x)$, where x is shifted gamma distributed, can be done using Gauss Laguerre integration by changing variables as follows

$$\int_{-\infty}^{\gamma} f(x) \frac{b^a}{\Gamma(a)} (\gamma - x)^{a-1} \exp(-b(\gamma - x)) dx = \int_0^{+\infty} f\left(\gamma - \frac{t}{b}\right) \frac{1}{\Gamma(a)} t^{a-1} \exp(-t) dt. \quad (9)$$

Generalized Gauss Laguerre quadrature is a numerical method to evaluate integrals of the form $\int_0^{\infty} f(x) dx$, where $f(x) = w(x)p(x)$ is the product of a weighting function $w(x)$ and a smooth function $p(x)$, in the sense of being well approximated by a polynomial. The weighting function is

$$w(x; s) = x^s \exp(-x), \quad (10)$$

where $s > -1$. Note that the classical Gauss Laguerre quadrature is the special case corresponding to $s = 0$.

Looking at (5) we see that in the shifted gamma model we have to set $s = a\rho - 1$ in the integration over the market factor. In Lévy base correlation model, in which we fix $a = 1$, this becomes

$$s = \rho - 1. \quad (11)$$

Note that it is crucial to apply generalized Gauss Laguerre quadrature and hence have the weighting function depend on the correlation parameter ρ . This can easily be seen as follows. In classical Gauss Laguerre quadrature the weighting function is $w(x; 0) = \exp(-x)$ and hence the factor $x^{\rho-1}$ is part of the function which is assumed to be smooth, in the sense of being well approximated by a polynomial. This is particularly troublesome for small values of the argument $x \rightarrow 0$, and increasing the number of integration points will not help much, as the classical Gauss Laguerre quadrature is designed to address the integration over the infinite interval. The need for the generalized method can be seen in elementary tests such as computing the moments of the gamma distribution. As an example, let us consider recovering default probabilities. In the generalized Gauss Laguerre quadrature the smooth function $p(x) = c$ is a constant and hence is integrated correctly by every formula, even with one single integration point. In the classical Gauss Laguerre quadrature the remainder function $p(x) = cx^{\rho-1}$ is not smooth for $x \rightarrow 0$ and even with a lot of integration points the constant c is not recovered accurately.

4 Computation of Abscissas and Weights

The disadvantage of applying the generalized Gauss Laguerre quadrature is that the weighting function depends on the correlation parameter ρ . Hence we have to be able to accurately and efficiently compute the abscissas and weights. Two algorithms are presented.

4.1 Eigenvalue Problem

The first algorithm is based on an eigenvalue problem (see Golub and Welsch [GW69] and Wilf [Wil62]). The abscissas of the quadrature formula can be found as the eigenvalues of the

symmetric tridiagonal matrix T , given by

$$T = \begin{pmatrix} \alpha_0 & \sqrt{\beta_1} & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{\beta_{n-2}} & \alpha_{n-2} & \sqrt{\beta_{n-1}} \\ & & & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{pmatrix} \quad (12)$$

where $\alpha_i = 2i + 1 + s$ and $\beta_i = i(i + s)$. The corresponding weights can be computed from the normalized eigenvectors. With

$$Tv_j = x_j v_j \quad (13)$$

where $v_j^T v_j = 1$ we have that

$$w_j = \mu_0 v_{j,1}^2, \quad (14)$$

where $\mu_0 = \int_0^{+\infty} w(x; s) = \Gamma(s + 1)$.

The symmetric tridiagonal eigenvalue problem as in (13) can be solved efficiently using the classical algorithms for eigenvalues; see Golub and Van Loan [GVL89], Parlett [Par80] or Wilkinson [Wil88] for details. This and other numerical algorithms are addressed in [PFTV92]. Note that only the first component of the normalized eigenvector is required, hence an optimized eigenvalue solver can be used.

4.2 Newton-Raphson Iteration

The second algorithm is based on the fact that good initial approximations for the zeros of the orthogonal polynomials are available (see Stroud and Secrest [SS66]). The initial guess \tilde{x}_1 for the first root x_1 is

$$\tilde{x}_1 = \frac{(1 + s)(3 + 0.92s)}{1 + 2.4n + 1.8s} \quad (15)$$

and for the second root x_2 it is

$$\tilde{x}_2 = x_1 + \frac{15 + 6.25s}{1 + 0.9s + 2.5n}. \quad (16)$$

For the other roots x_i , the initial guess \tilde{x}_i is

$$\tilde{x}_i = x_{i-1} + \left(\frac{1 + 2.55(i-2)}{1.9(i-2)} + \frac{1.26(i-2)s}{1 + 3.5(i-2)} \right) \frac{x_{i-1} - x_{i-2}}{1 + 0.3s}. \quad (17)$$

With these initial guesses a Newton-Raphson iteration is started, that is subsequent approximations to the root $f(x_i) = 0$ are constructed as follows

$$x_i^{(k+1)} = x_i^{(k)} - \frac{f(x_i^{(k)})}{f'(x_i^{(k)})} \quad (18)$$

where $f'(x)$ denotes the derivative of the function $f(x)$ with respect to x . Since Newton-Raphson iterations converge quadratically, provided a good starting value is given, we have very accurate approximations to the root after a few iterations.

The polynomials can be evaluated efficiently using the recurrence relation

$$(n+1)L_{n+1}^{(s)}(x) = (2n+1+s-x)L_n^{(s)}(x) - (n+s)L_{n-1}^{(s)}(x), \quad (19)$$

which can be started with $L_0^{(s)}(x) = 1$ and $L_{-1}^{(s)}(x) = 0$. The derivative is easily obtained from the following relation

$$x \frac{d}{dx} L_n^{(s)}(x) = nL_n^{(s)}(x) - (n+s)L_{n-1}^{(s)}(x). \quad (20)$$

For these and other formulas involving Laguerre polynomials, see e.g. Abramowitz and Stegun [AS64]. The weights are given by

$$w_i = \frac{-\Gamma(s+n)}{n\Gamma(n)L_{n-1}^{(s)}(x_i) \frac{dL_n^{(s)}}{dx}(x_i)}. \quad (21)$$

Note that we only have to evaluate the gamma function once. Given $\Gamma(s+1)$, we can make use of the recurrence relation $\Gamma(x) = (x-1)\Gamma(x-1)$ for other values of n .

5 Conclusions

In this paper we described the need for Generalized Gauss Laguerre integration in the Lévy Base Correlation algorithm. The disadvantage is obvious: the quadrature formula depends on the base correlation parameter and consequently there is computational overhead in generating the quadrature formula. However accuracy considerations clearly rule out classical tabulated Gauss Laguerre quadrature. The algorithms presented in here are efficient in computing the Generalized Gauss Laguerre quadrature formulas, making the Lévy Base Correlation model a very tractable algorithm.

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