One Credit Event Models for CDOs of ABS

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Abstract

In this paper we look at a multifactor Monte Carlo Gaussian Copula based model to price CDO's of ABS's. The probabilities of default are implied from prices of ABS bonds and several notional amortisation schedules are proposed. A detailed sensitivity analysis is done with respect to recovery rates, default intensities, amortization schedules and risky duration, for all the individual bonds. Additionally a similar analysis is done of the impact of the parameters and the amortization schedules on the prices of the different tranches of a CDO of ABS's.

1 Introduction

The term *securitization* refers to the pooling and packaging of financial assets in the form new securities that are sold to investors. Via securitization financial institutions create instruments that can be sold into the market instead of kept in their balance sheets. On the issuer side it improves leverage ratios, the efficient use of capital, and lowers the cost of funding. Additionally it permits the instutitions to focus on the business side instead of managing the assets in the balance sheet.

This was the securitization business model until June 2007 when the credit crunch brought up additional awareness for the consequences underlying the business model. On one side the low costs associated with the activity has brought tremendous economic growth lowering the costs of good and services while making them more affordable to a larger portion of the population. On the other side however by substituting a single name instrument (say a single name bond) by a securitization note (say a CDO note) the investor inherently substitutes idiosyncratic by for systematic risk.

From an investor point of view the investment on a pool of assets inherently implies in diversification. The diversification however comes at a price: as experienced during the referred credit

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crunch the instrument increases the level of *systematic* risk to the portfolio implying a higher cost of regulatory capital than the one taken so far by regulatory capital. In order to be able to foster the securitization business model at low cost of capital one needs to put the securities in mark to market while managng the risks via the standardised credit indices. This implies in transparency on methodologies for *pricing* purposes (for evaluation of hedge paremeters) and the hability to map portions of the bepoke portfolio into the capital structure of standardised credit indices. For the management of the credit portfolios of banks in a way to suport the securitization business model of a financial institution we refer to Garcia, Goossens and Lamoot [GGL08], for a proposal to adapting the standard algorithm to price TABX (the standadised credit index tranch for the index of CDS on subprime MBS's) we refer to Garcia and Goossens [**?**]. In order to use the standard methodology (used to price the standardised credit indices) to price bespoke portfolios a widely used approach is called *correlation mapping* and we refer to Garcia and Goossens [GG08b] for the use of such an approach. In order to use the TABX approch to price CDO's of ABS's one needs to have risk free

In order to referred TABX algorithm in the most appropriate fashion it is necessary the existence of liquid ABCDS spreads, risk neutral market accepted assumptions for prepayment, and available correlation figures for pricing tranches on the indices. Given the absence of such a liquid market instrument with known input parameters an alternative approach is to price the CDO of ABS's using different approaches while stressing some input parameters. In this note we compare available algorithms to price CDO's of ABS's using MC simulation under the gaussian copula. This is done using the 1-factor market approach under two different correlation assumptions and a multifactor model using an approach developed by Derivative Fitch [ZKG+07] for Structured Finance CDO's. Additionally we give the impact of different assumptions for the amortization profiles.

The paper willbe organised as follows. In section 2 we show the approach used price an ABS bond, we show different assumptions for the amortization parameter and show how to use it to imply default probabilities. In section 3 we analyse the impact of the different model parametes for ABS pricing. In section 4 we give a brief description of the multifactor gaussian copula approach inspired in the Derivative Fitch model. In section 5 we show how to get default times from simulated returns. In section 6 we show the results of of the simulations for the pricing of different tranches of the CDO of ABS. In section 7 we give the conclusions.

2 ABS Bond and ABCDS

In this section we give valuation formulas for an ABS bond and its CDS. Observe that the main difference between an ABS with respect to a simpler corporate bond is that due to amortization and the real possibility of prepayments the outstanding notional is time-dependent. In what follows we will make some simplifying assumptions. First, we assume the amortized (ABS) bond is a floater paying a coupon C above Libor. Second, the amortization schedule is deterministic. Third, we apply a one default event model, that is we assume the ABCDS behaves as a corporate bond, that is in case of default the ABCDS stops existing and the protection seller receives the recovery (R) value of the ABS. Although this is current market practice we note that this is a very strong assumption.

In what follows we will assume four amortization profiles:

- Conditional Prepayment Rate (CPR)

$$n_t = (1-c)^t \tag{1}$$

- Bullet

$$n_t = \begin{cases} 1 & t \leq T_b \\ 0 & t > T_b \end{cases} \tag{2}$$

- Linear

$$n_t = \begin{cases} 1 - \frac{t}{T_l} & t \leq T_l \\ 0 & t > T_l \end{cases} \tag{3}$$

- Quadratic

$$n_t = \begin{cases} 1 - \left(\frac{t}{T_q}\right)^2 & t \leq T_q \\ 0 & t > T_q \end{cases} \tag{4}$$

Consider there are N payment dates, n_i is the outstanding notional at payment date t_i, the average life (AL) of the ABS is given by:

$$AL = \sum_{i=2}^{N} (n_{i-1} - n_i)\, t_i \tag{5}$$

and we assume that $t_1 = 0.0$, $t_N = T$ (maturity of the ABS), $n_1 = 1.0$ and $n_N = 0.0$. A first step in the pricing algorithm of an ABS note is to calculate the amortized notional using a prepayment assumption. The prepayment parameter is calibrated in such a way that one recovers observed average life.

We assume the default process to follow a homogeneous Poisson process and as such for any $0 \leq t \leq T$ the risk neutral survival probability $q(t_i)$ at time t_i and the default intensity λ, satisfy:

$$q_i = q(t_i) = \exp(-\lambda t_i). \tag{6}$$

We assume a risk free discount rate (r) given by a flat interest rate curve:

$$d_i = d(t_i) = \exp(-r t_i) \tag{7}$$

The price of the amortized (ABS) bond B is computed as

$$\begin{aligned} B &= \sum_{i=2}^{N} n_{i-1}(L_i + c) d(t_i) q(t_i) \Delta t_i + q(t_N) d(t_N) n_N \\ &+ \sum_{i=2}^{N} (n_{i-1} - n_i)\, d(t_i) q(t_i) + R \sum_{i=2}^{N} n_{i-1} d(t_i) \left(q(t_{i-1}) - q(t_i)\right). \end{aligned} \tag{8}$$

In these equations the summations run over the payment dates t_i and L_i is the Libor rate at time t_i. The recovery rate is R, and $\Delta t_i = t_i - t_{i-1}$ is the year fraction. In the present study for simplicity we considered $L_i = r$.

Additionally two important quantities are *risk duration* (D_R) and *expected loss* ($\mathbb{E}L$) computed respectively as

$$D_R = \sum_{i=2}^{N} n_{i-1} d_i q_i \left(t_i - t_{i-1}\right) \tag{9}$$

$$\mathbb{E}L = (1-R) \sum_{i=2}^{N} n_{i-1} d(t_i) \left(q(t_{i-1}) - q(t_i)\right) \tag{10}$$

The fair spread s on the deal is given by

$$s = \frac{\mathbb{E}L}{D_R} \tag{11}$$

In the next section we show the results of the algorithm here used and the sensitivities to the input parameters.

3 Single Name Sensitivity

In the first step of the pricing algorithm for the ABS bond we calibrate the appropriate parameters in the chosen prepayment assumption (see 2) to match the observed AL (see eq. 5). The output of this step is the amortised notional ($n_i's$ for $2 \le i \le N$). In the second step one uses the price of the bond (see eq. 8) to imply default probabilities, and in the case of a homogeneous Poisson process (see eq. 6) this means a suitable intensity $lambda$.

In the results shown in this section we assume the ABS has a coupon $c = 55$ bp and the interest rate flat at ($r =$) 2.6881%. The average life is ($AL =$) 8 years, the bond price is ($B =$)0.80 and the maturity is ($T =$)30 years. We assume quarterly payments, $\Delta t = (t_i - t_{i-1} =)$ 0.25, and hence $N = 121$. The four prepayment assumptions used in this study are shown in Fig. 1.

In Fig. 1 we show the variation of the outstanding notional with respect to the different amortization profiles assuming the same AL for all the amortization schedules. As observed in the figure the order of decreasing amortization speed goes from *cpr*, *linear*, *quadratic* and finally *bullet*.

In fig. 2 we see the impact of the recovery rate assumptions on the expected loss for the different amortization profiles. As expected for any amortization profile a higher recovery rate implies a lower expected loss. There is a cap on the input recovery rate to be compatible to observed market prices. As an example if recovery rate is 90% one can not have a price below 90 cents. As it is also natural the figure shows that for a certain recovery rate the earlier prepayment occurs the lower the expected loss.

In fig. 3 we see the impact of the recovery rate assumptions on the fair spreads with for the different amortization profiles. For a certain amortization profile the higher the recovery assumption the higher should be the spreads in order to justify observed market prices. Another

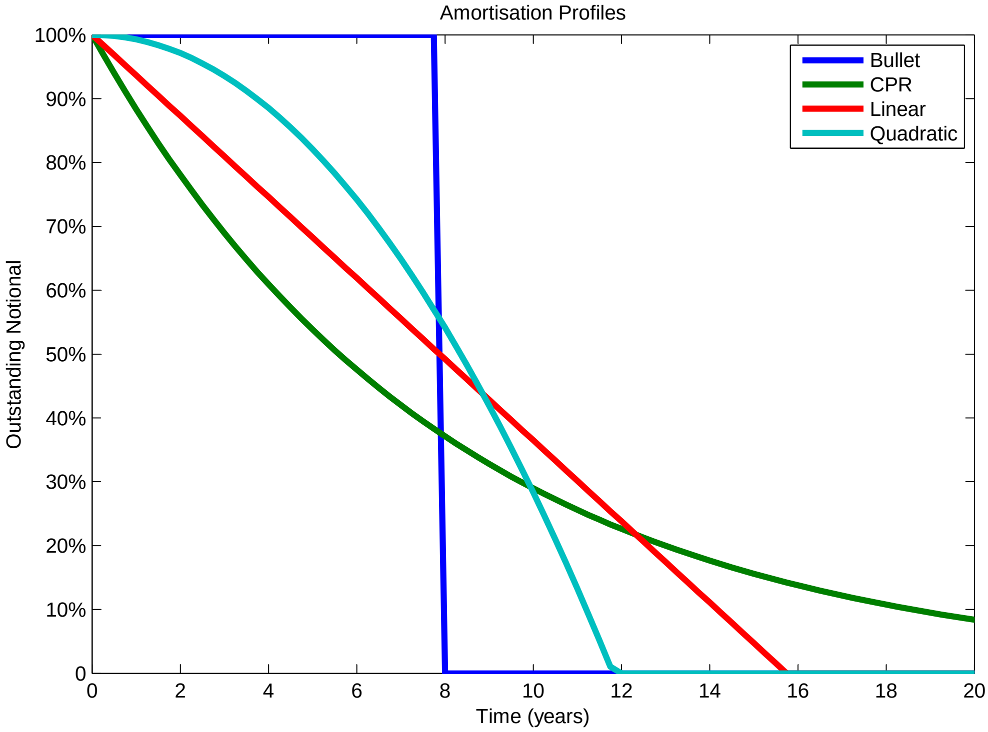

Figure 1: Variation of the Outstanding Notionals in Time for the Different Amortisation Profiles

consequence of the constant price is that for a given recovery rate the amortization profiles with higher prepayment assumptions at the begining will have higher spreads.

In fig. 4 we see the impact of the recovery rate assumptions on the risky duration for the different amortization profiles. For a certain amortization profile the higher the recovery the higher the spreads and consequent decrease in risk duration. If one fixes the recovery rate the higher the amortization profile in the begining the lower the risk duration.

In fig. 5 we see the impact on the default intensity with respect to recovery rate assumptions for the different amortization profiles. For a certain amortization profile the higher the recovery the higher the default intensity in order to justify observed market prices. Additionally the earlier one prepays the higher the default component on the price and as such the higher the default intensity.

The results of the pricing algorithm (to match the observed bond price and average life) with respect to different assumptions of prepayment functions are shown in tables 3 and 3 for respectivelly 0% and 40% recovery rates. It serves as a more detailed summary of what has been observed in the figures so far mentioned.

For a certain amortization profile decreasing the recovery rate increases the expected loss and

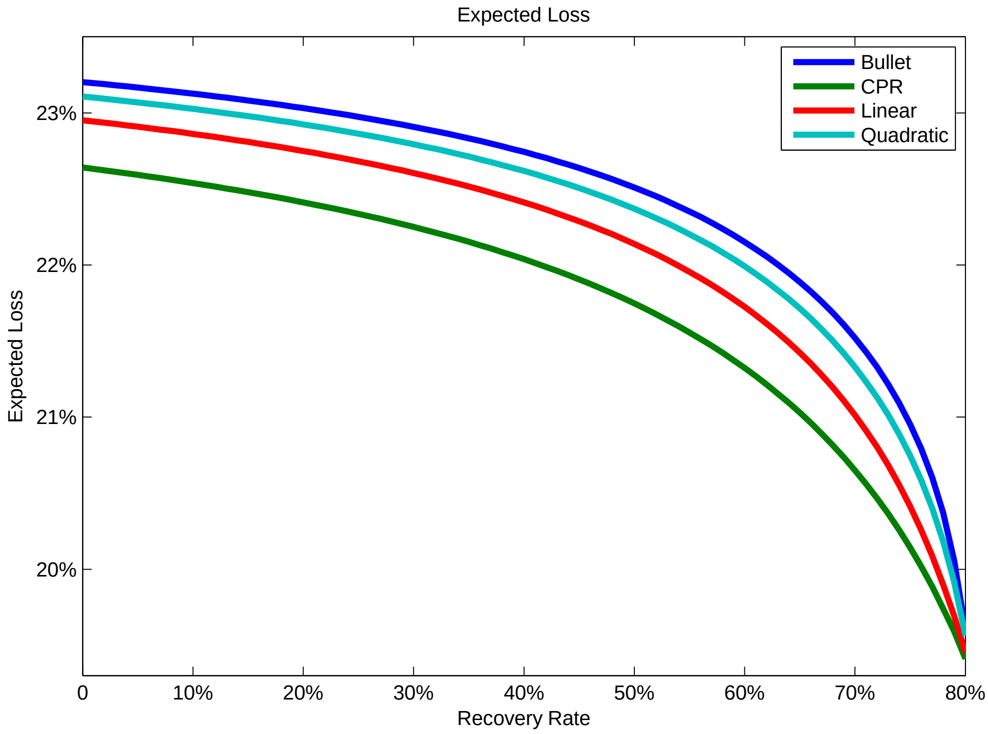

Figure 2: Expected Loss with respect to Recovery rate Assumptions for the different Amortization Profiles

the risk duration. As prices are kept fixed it implies lower spreads and default intensities. The earlier prepayment happens with lower risk duration the higher one needs for the default intensity and as such the higher spread to justify the same bond prices.

In the next sections we show how to use a Monte Carlo algorithm to price CDO's of ABS.

4 Multi Factor Correlation Model

In *latent variable* models default occurs when a certain variable Y_i falls below a threshold K_i which is implied from CDS prices. The so called *market* or *systemic* factor X and the *idiosyncratic* factor $X^{(i)}$ are random variables whose functional form depends on model assumptions.

In the generic one factor Gaussian model the latent variable is represented as

$$Y_i = \rho X_m + \sqrt{(1-\rho^2)}\xi_i \tag{12}$$

where X_m and ξ_i are the market and the idiosyncratic factors, independent and identically distributed variables. Both have the same distribution $N(0,1)$.

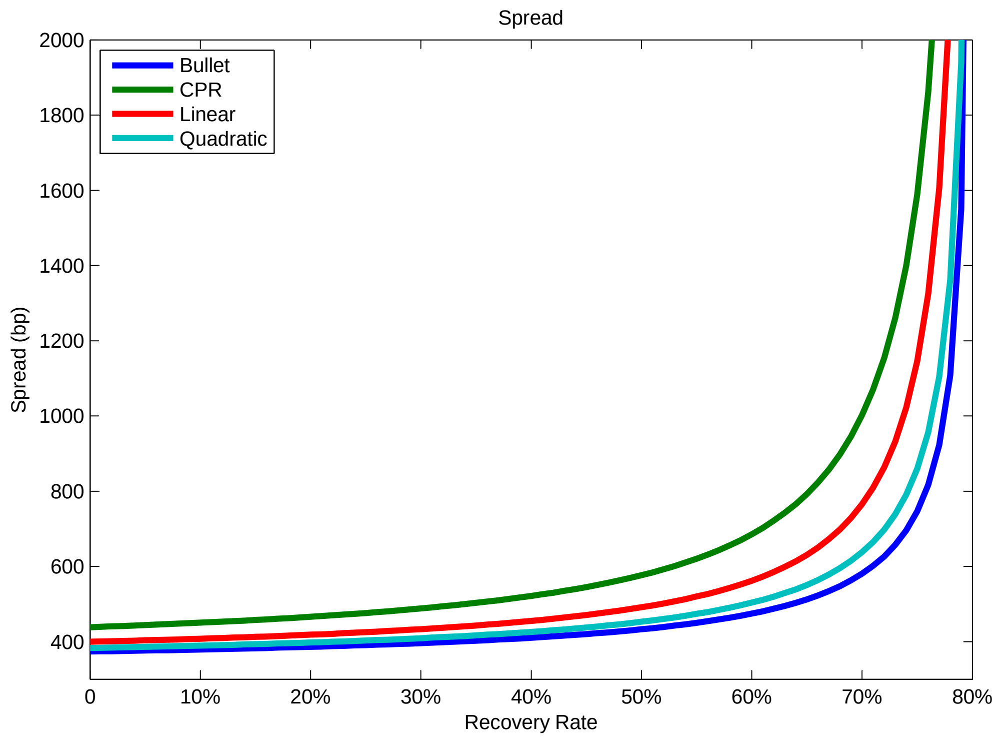

Figure 3: Fair Spreads with respect to Recovery rate Assumptions for the different Amortization Profiles

In our case however we extend the simple 1-factor model to a multifactor model and in this case the latent variable for a single entry is modeled as follows

$$Y_i = \xi_g X_g + \xi_r X_r + \xi_s X_s + \xi_4 X_4 + \xi_5 X_5 + \xi_6 X_6 + \xi_7 X_7 + \xi_8 X_8 + \xi_i X_i, \tag{13}$$

where all variates X are standard normal distributed. The coefficient $\xi_g = \sqrt{0.45}$ determines the global correlation. Other ξ coefficients are zero, except $\xi_r = \sqrt{0.10}$ for subprime RMBS or $\xi_s = \sqrt{0.10}$ for structured finance CDO and $\xi_4 = \sqrt{0.05}$ if the vintage is 2004, or $\xi_5 = \sqrt{0.10}$ if the vintage is 2005, or $\xi_6 = \sqrt{0.20}$ if the vintage is 2006, or $\xi_7 = \sqrt{0.20}$ if the vintage is 2007, or $\xi_8 = \sqrt{0.20}$ if the vintage is 2008.

The coefficient for the idiosyncratic remainder is

$$\xi_i = \sqrt{1 - \xi_g^2 - \xi_r^2 - \xi_s^2 - \xi_4^2 - \xi_5^2 - \xi_6^2 - \xi_7^2 - \xi_8^2}. \tag{14}$$

This is the approach used by Ficth for their CDO's of ABS's *before* the credit crunch that began in Jun 2007. In this case the maximum correlation possible reaches 75%. To our knowledge there is no detailed documentation available for any update on the methodology made after the credit crunch.

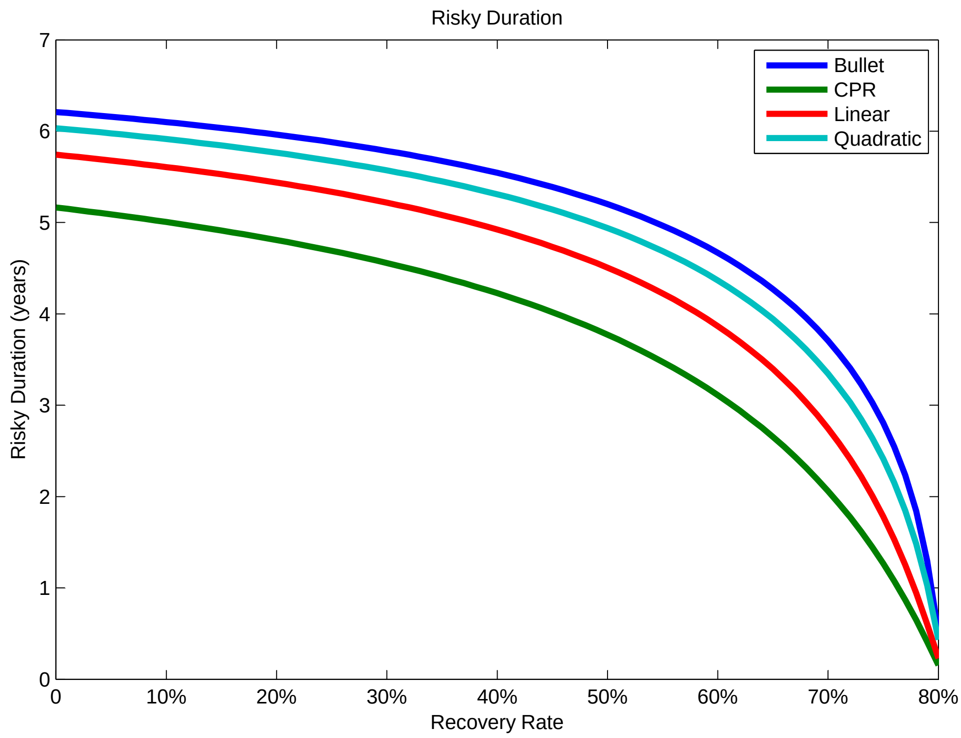

Figure 4: Risky Durations with respect to Recovery rate Assumptions for the different Amortization Profiles

5 Monte Carlo Simulation

Tipically in a MC simulation one generates random numbers for the market and idiosyncratic factors that are plugged in eq. and eq. 14 to generate the return Y_i. With the value of the default intensity (λ_i) implied from ABS bond prices (see eq. 6 in section 2) one determines the default time τ_i as

$$\tau_i = \frac{-1}{\lambda_i} \log(1 - N(Y_i)), \tag{15}$$

where $N(x)$ is the standard normal cumulative distribution. If the default time of a certain ABS in the portfolio is lower than its maturity date then the contract is terminated and the outstanding notional is substituted by its recovery value. The contract will go until maturity otherwise.

Each simulation consists of generating one set of systematic factors and as many idiosyncratic factors as the number of ABS's in the portfolio. The default times of all the underlying collateral are determined and in so doing one can determine the present value of loss side.

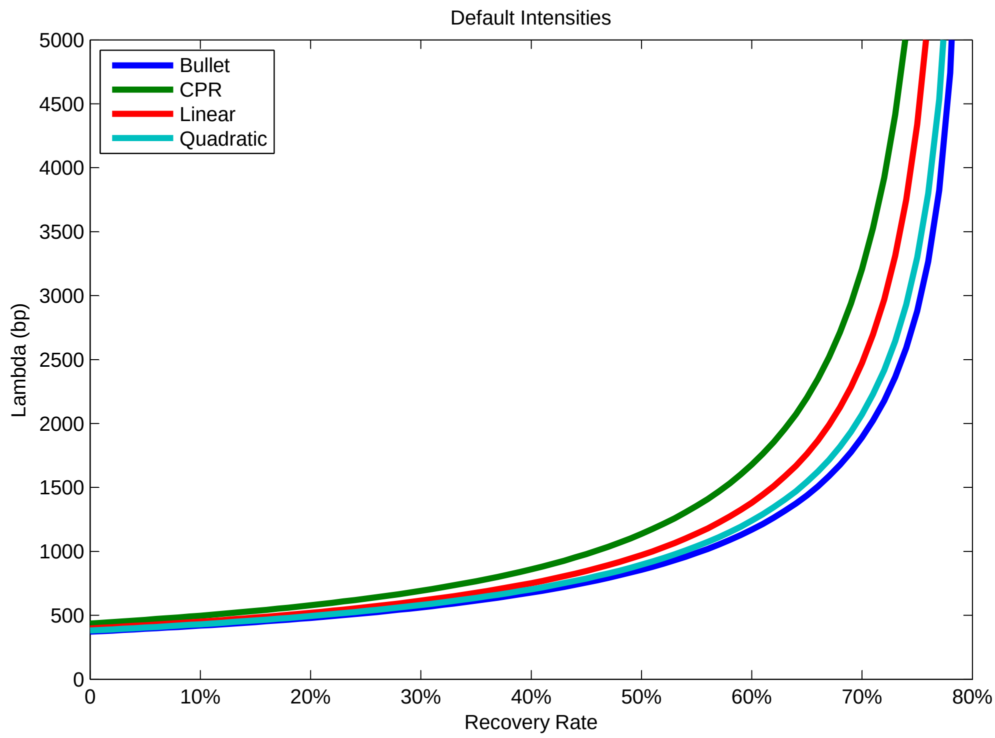

Figure 5: Default Intensitieswith respect to Recovery rate Assumptions for the different Amortization Profiles

6 Results

In this section we show the results of the MC simulation proceedure both 1-factor and multifactor for the different tranches of a CDO of ABS's under different assumptions of amortization schedules and recovery rates. As already mentioned in section 4 the multifactor simulation uses the correlation structure of Fitch for CDO's of structured finance ([ZKG+07]), and for the 1-factor model we show the results using flat 75% and 90% correlations. The amortization assumptions are CPR, Bullet, Linear and Quadratic and have been described in more detail in section 2. The different recovery rate assumptions are 0%, 40%, 50%, 60% and 70%, flat for all ABS's, and Moody's rating agency recovery rate (with a few minor corrections if the bond prices turns out to be lower than the recovery rates) specific for each ABS in the collateral portfolio.

The results are shown in the table 3 in terms of expected loss divided by tranch sizes. The tranches in question are [0,0.10%], [0.10%,3.10%], [3.10%,10.0%] and [10.0%,100.0%] and on the table they are represented by their attachment points. The table is subdivided in four subtables characterized by the amortization schedule used to generate the results. The upper left subtable

amortisation	$\mathbb{E}$Loss (%)	RD	λ (bp)	spread (bp)
bullet	23.2024	6.2093	371.9386	373.6732
cpr	22.6410	5.1644	436.0220	438.4071
linear	22.9519	5.7429	397.6705	399.6538
qaudratic	23.1076	6.0328	381.2126	383.0350

Table 1: ABS characteristics for different assumptions of prepayment and with recovery rate assumed zero ($R = 0$)

amortisation	$\mathbb{E}$Loss (%)	RD	λ (bp)	spread (bp)
bullet	22.7432	5.5437	677.9809	410.2555
cpr	22.0384	4.2261	859.8282	521.4817
linear	22.4113	4.9232	751.5958	455.2209
quadratic	22.6180	5.3096	703.7445	425.9830

Table 2: ABS characteristics for different assumptions of prepayment and with recovery rate set at 40% ($R = 40\%$)

reffers to the Bullet, the lower left to the Linear, the upper right to the CPR and the lower right to the Quadratic amortization schedlue. Each subtable belonging to a certain ammortization schedule contains three other tables (upper, mid, down) characterized by the type of the MC simulation made. The upper and mid tables refer to a 1-factor MC simulation with 90% and 75% flat correlations, while the lower refers to the multifactor MC using the Ficth correlation function.

Within the same amortization profile comparing the upper and the mid tables for a given recovery rate the losses of the equity (senior) tranches decrease (increase) with higher flat correlation. For the case of the multifactor model the maximum value of the correlation is 75% indicating the in general the correlation level is lower than the 75% flat for the 1-factor model. This means that indeed for a given recovery rate the equity (senior) tranch will have higher (lower) losses than for the case of the 1-factor model. For a given amortization profile and within the same simulation approach we observe that as the bond prices are fixed unless the recovery rate reaches a too high value incompatible with bond prices we know that the total loss is independent of the recovery rate. It implies that increasing recovery rates will shift the losses from the senior tranches to the equity tranches.

As it would be expected from fig. 1 the impact of the different amortization profiles on the expected loss of the tranches in increasing (decreasing) order of losses for the senior (equity) tranch is the following: CPR, Linear, Quadratic and Bullet. Observe that the impact of the amortization profiles on the supersenior tranch is huge. As an example using the rating agency recovery rate there is a decrease of 25% when one moves from Bullet to CPR amortization assumption and the multi factor model. Observe that the non-arbitrage conditions are satisfied (see Garcia and Goossens [GG08a]). In particular note that the relative loss that is the loss divided by tranch size decreases as one moves up the capital structure.

Bullet: 3.40% CPR:2.55% Linear 2.90% Quadratic 3.21%

7 Conclusions

In this paper we have outlined a traditional Gaussian Copula based simulation model to price tranches of a CDO of ABS's. For each individual ABS in the collateral the probability of default is implied from observed ABS bond price with a flat default intensity. Results have been shown for four different assumptions of amortization profiles that are each calibrated to market observed average lives of the ABS's. Detailed analysis has been done for the sensitivity of the expected loss, spreads, risky duration and default intensity for different assumptions of recovery rates and amortization profiles. Additionally the same sort of detailed sensitivity studies has been done for the pricing of tranches of a CDO of ABS.

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0.00%	0.10%	3.10%	10.00%	A	0.00%	0.10%	3.10%	10.00%
42.29%	35.00%	25.74%	8.45%	0	75.64%	49.90%	32.69%	7.10%
59.32%	47.70%	34.92%	7.17%	40%	89.05%	64.54%	42.01%	5.68%
66.78%	53.28%	38.77%	6.61%	50%	92.70%	70.40%	45.88%	5.14%
76.43%	60.72%	44.04%	5.83%	60%	95.90%	77.52%	50.82%	4.39%
90.56%	72.27%	51.42%	4.62%	70%	98.53%	86.61%	57.39%	3.40%
94.88%	74.39%	49.01%	4.61%	RA	98.66%	86.65%	54.89%	3.41%
58.13%	47.49%	32.34%	7.53%	0	84.29%	59.46%	37.86%	6.45%
73.58%	59.18%	40.41%	6.37%	40%	93.26%	71.94%	46.20%	5.15%
79.45%	64.04%	43.85%	5.88%	50%	95.40%	76.67%	49.85%	4.66%
86.21%	70.22%	48.50%	5.13%	60%	97.29%	82.48%	54.28%	3.96%
94.50%	79.28%	55.24%	4.09%	70%	98.87%	89.82%	60.55%	3.06%
96.66%	80.30%	53.21%	4.09%	RA	98.95%	89.74%	58.27%	3.08%
78.22%	63.87%	41.34%	6.26%	0	93.29%	72.82%	45.41%	5.38%
89.20%	73.72%	48.38%	5.20%	40%	97.08%	82.19%	53.13%	4.31%
92.31%	77.36%	51.33%	4.79%	50%	97.89%	85.49%	56.08%	3.86%
95.40%	81.85%	55.26%	4.22%	60%	98.60%	89.33%	60.01%	3.29%
98.14%	87.88%	61.01%	3.38%	70%	99.29%	93.82%	65.55%	2.49%
98.75%	88.41%	58.56%	3.40%	RA	99.39%	93.82%	62.65%	2.55%

0.00%	0.10%	3.10%	10.00%	A	0.00%	0.10%	3.10%	10.00%
58.92%	42.89%	29.90%	7.82%	0	51.27%	38.42%	27.43%	8.22%
76.10%	56.82%	39.10%	6.38%	40%	68.75%	51.59%	36.38%	6.81%
82.69%	62.98%	43.05%	5.75%	50%	75.64%	57.59%	40.43%	6.25%
90.09%	71.05%	48.20%	4.95%	60%	83.89%	65.15%	45.49%	5.44%
97.30%	81.97%	55.24%	3.83%	70%	94.71%	76.60%	52.71%	4.29%
98.11%	82.05%	52.33%	3.85%	RA	97.04%	77.75%	50.32%	4.33%
73.10%	53.95%	35.35%	7.05%	0	66.65%	50.37%	33.54%	7.34%
86.15%	66.28%	43.83%	5.65%	40%	80.98%	62.07%	41.55%	6.06%
90.29%	71.11%	47.25%	5.15%	50%	85.96%	67.16%	45.23%	5.57%
94.49%	77.62%	51.92%	4.44%	60%	91.15%	73.37%	49.60%	4.85%
98.25%	86.30%	58.63%	3.44%	70%	96.87%	82.23%	56.38%	3.85%
98.56%	86.26%	56.45%	3.46%	RA	97.90%	82.81%	54.36%	3.84%
88.16%	68.97%	43.59%	5.81%	0	84.40%	66.19%	42.10%	6.04%
94.97%	78.59%	51.10%	4.71%	40%	92.91%	75.97%	49.48%	5.01%
96.59%	82.22%	54.29%	4.28%	50%	95.08%	79.42%	52.34%	4.59%
97.97%	86.39%	58.13%	3.67%	60%	97.09%	83.86%	56.30%	4.00%
99.06%	91.85%	63.99%	2.85%	70%	98.75%	89.62%	62.15%	3.16%
99.21%	91.94%	61.45%	2.90%	RA	99.01%	89.98%	59.68%	3.21%

Table 3: Expected loss for the different tranches of a CDO of ABS's for different amortization profiles (CPR, Bullet, Linear and Quadratic (text for details) and different recovery rates assumptions (0%, 40%, 50%, 60%, 70% and deal especific Rating Agency recovery rate. The upper left table is the