Lévy Base Correlation Explained

João Garcia*† Serge Goossens‡§

First version: June 23, 2007
This version: August 12, 2007

Abstract

In an earlier paper we introduced Lévy base correlation. In this paper we compare the Gaussian copula and Lévy base correlation models. The results of a historical study of both models on the iTraxx Europe Main dataset are presented. We focus on the base correlation surface and on the deltas of the tranches with respect to the index.

1 Introduction

There has been an enormous technological innovation behind the incredible growth of the credit derivatives market. The success of the credit derivatives instruments can be seen by the growth in the outstanding notional amount in the market from about 3,000 billion USD back in 2003 to 20,000 billion USD in 2006 (see e.g. [Ass06]).

Since the introduction of the 1-factor Gaussian copula model for pricing synthetic Collateralized Debt Obligation (CDO) tranches by Andersen et al. [ASB03] correlation is seen as an exogenous parameter used to match observed market quotes. First the market adopted the concept of *implied compound correlation*. In tandem with the concept of volatility in the Black Scholes option pricing framework compound correlation was the parameter to be put in the model to match observed market prices of tranches.

One of the problems of this approach resides in its unsuitability for interpolation. Given the implied compound correlations for the [3%–6%] and [6%–9%] tranches of a liquid index such as iTraxx Europe Main[1] it is not clear which value to use for a nonstandard tranche such as e.g.

*Dexia Group, Rogierplein 11, B1210 Brussels, Belgium (crisj@dexia.com)

†The authors would like to thank Alex Van Tuykom (alex.vantuykom@dexia.com), Global Head of Modelling at Financial Markets, Dexia Holding, for supporting this work and enabling its publication.

‡Dexia Bank, Pachecolaan 44, B1000 Brussels, Belgium (Serge.Goossens@dexia.be)

§The opinions expressed in this paper are those of the authors and do not necessarily reflect those of their employers.

[1] The standard attachment points are 3%, 6%, 9%, 12% and 22% for iTraxx Europe Main and 3%, 7%, 10%, 15% and 30% for CDX.NA.IG.

[5%–8%]. Besides the problems of interpolation we can also mention that during some market events correlation may receive non-meaningful values (e.g. during the auto crisis of May 2005).

The current widespread market approach is to use the concept of base correlation (BC) introduced by McGinty et al. [MBAW04]. In the base correlation methodology only equity or base tranches[2] are defined. The price of a tranche [A – D] is calculated using the two equity tranches with A and D as detachment points. Using BC it is quite straightforward to bootstrap between standard attachment points. Additionally the BC concept is quite adapted to interpolation for nonstandard tranches. Hence the [5%–8%] tranche would be priced by interpolating the BC curve for values at 5% and 8% respectively.

The methodology however has some weaknesses. First of all it is very sensitive to the interpolation technique used. Even worse, the methodology may not be arbitrage-free. Finally the methodology does not provide any guidance on how to extrapolate the curve, especially below the 3% attachment point. In an earlier paper we [GG07] have addressed these problems and we have shown how to perform interpolation in the base correlation framework that avoids arbitrage.

Another characteristic of the standard market approach is its reliance on the Gaussian copula. In an earlier paper [GGMS07] we have introduced the concept of Lévy base correlation. In this paper we present the results of a historical study of both models on the iTraxx Europe Main dataset.

The remainder of this paper is organised as follows. In section 2 we review the generic 1-factor model for valuation of CDO tranches. Our Lévy base correlation methodology is detailed in section 3. The historical study is outlined in section 4. We look at base correlation in these two models in section 5. First we show the evolution over time and second we consider the behaviour across maturity and look at the base correlation surface. In section 6 we compare hedge parameters in the different models. We focus on the deltas of the tranches with respect to the index. This is also done across maturity. Finally our conclusions are presented in section 7.

2 Generic 1-Factor Model

The 1-factor Gaussian copula model using the so called *recursion algorithm* was first introduced by Andersen et al. [ASB03] and is in widespread use by market participants. In what follows we give a brief description of the algorithm and we refer to Andersen and Sidenius [AS05] for an excellent survey on the subject.

2.1 CDO Tranche Valuation

Consider a portfolio of N firms and fix a time horizon T. It is standard market practice to assume the default process to follow an inhomogeneous Poisson process and as such for any $0 \leq t \leq T$ the default times τ_i and default intensities $\lambda_i(t)$, $i = 1, \ldots, N$, satisfy

$$\mathbb{P}(\tau_i > t) = \exp\left(-\int_0^t \lambda_i(u)\mathrm{d}u\right) \tag{1}$$

[2]Base or equity tranches are tranches with attachment point 0.

where $\mathbb{P}$ is the risk-neutral probability measure. In a 1-factor model of portfolio defaults, a single systemic factor X is introduced, conditional upon which all default probabilities are independent. The single name survival probabilities $\mathbb{P}(\tau_i > t)$ are typically implied from the credit default swap (CDS) market.

The fair spread of a CDS balances the present value of the contingent leg C, given by

$$C = N^{\text{CDS}}(1-R)\sum_{i=1}^{n} d(t_i)\left(P_S(t_{i-1}) - P_S(t_i)\right) \tag{2}$$

and the present values of the fee leg F, given by

$$F = N^{\text{CDS}} S\left(\sum_{i=1}^{n} P_S(t_i) d(t_i)\Delta t_i + A_D\right), \tag{3}$$

where N^{CDS} is the CDS notional and A_D is the accrual on default

$$A_D = \frac{1}{2}\sum_{i=1}^{n} N^{CDS} d(t_i)\left(P_S(t_{i-1}) - P_S(t_i)\right)\Delta t_i. \tag{4}$$

In these equations the summations run over the payment dates, S is the spread premium on a yearly basis, $P_S(t_i)$ is the survival probability at time t_i, R is the recovery rate, $d(t)$ is the risk-free discount factor and $\Delta t_i = t_i - t_{i-1}$ is the year fraction.

The key step in valuing CDO tranches is to compute the joint loss distribution. In the recursion algorithm one computes a discretised version of the conditional loss distribution by means of a simple recursion formula. A loss unit u is chosen so that within a certain tolerance, losses can be represented by integers. For the iTraxx Europe Main portfolio with an assumed uniform recovery rate of 40%, the loss unit is 0.48%. We denote by $P^{(i)}(l, t|X)$ the probability of l losses (in terms of the loss unit u) at time t with i names conditional on the market factor X. Recalling that conditional on X all default probabilities are independent, we can write that $P^{(i)}(l, t|X)$ is the sum of two terms:

$$P^{(i)}(l, t|X) = P^{(i-1)}(l, t|X)\mathbb{P}(\tau_i > t|X) + P^{(i-1)}(l - \omega_{(i)}, t|X)(1 - \mathbb{P}(\tau_i > t|X)), \tag{5}$$

where $\omega_{(i)}$ is the number of loss units incurred by a default of the ith name. The unconditional loss distribution is found by integrating over the market factor

$$P(l, t) = \int_{\Omega_X} P(l, t|X) f(X)\mathrm{d}X, \tag{6}$$

where $f(X)$ is the density of the probability distribution of the market factor X. Analogous to the CDS case the fair spread of a CDO tranche balances the present value of the fee leg F, given by

$$F = S\sum_{i=1}^{n}(N^{(\text{Tr})} - \mathbb{E}[L_i^{(\text{Tr})}])d(t_i)\Delta t_i \tag{7}$$

and the present value of the contingent leg C, given by

$$C = \sum_{i=1}^{n} d(t_{i+\frac{1}{2}}) \left(\mathbb{E}[L_i^{\text{(Tr)}}] - \mathbb{E}[L_{i-1}^{\text{(Tr)}}] \right). \tag{8}$$

In these equations the summations run over the payment dates, S is the spread premium on a yearly basis, $d(t)$ is the risk-free discount factor, $\Delta t_i = t_i - t_{i-1}$ is the year fraction, $\mathbb{E}[L_i^{\text{(Tr)}}]$ is the expected loss on the tranche at time t_i and $N^{\text{(Tr)}}$ is the tranche size.[3] In the base correlation framework, the expected loss on a tranche is computed as the difference of the expected loss of two equity tranches

$$\mathbb{E}L[\text{A–D}] = \mathbb{E}L[\text{0–D}; \rho_D] - \mathbb{E}L[\text{0–A}; \rho_A].$$

For more details on base correlation we refer to McGinty et al. [MBAW04] and to O'Kane and Livasey [OL04].

Other approaches to compute the joint loss distribution exist. We refer to Moody's [DSFT03] for models based on Fast Fourier Transform (FFT) and to Yang et al. [YHZ06] for models based on the Saddle Point Approximation.

2.2 Latent Variable Models

In the so called *latent variable* model default occurs when a certain (latent) variable A_i (usually the return) falls below a certain threshold K_i that is implied from CDS prices.[4] The *market* or *systemic* factor X and the *idiosyncratic* factor $X^{(i)}$ are random variables whose functional form depends on model assumptions.

2.3 Generic One-Factor Lévy Model

This section is a short summary of a few sections in our earlier paper [GGMS07]. For more details on Lévy processes we refer to Bertoin [Ber96] and Sato [Sat00]. For the sake of notations we limit ourselves to the definitions that follow.

If for every positive integer n, the characteristic function $\phi(u) = \mathbb{E}[\mathrm{i}uX]$ is also the nth power of a characteristic function, the distribution is said to be *infinitely divisible*. Given an infinitely divisible distribution, a stochastic process, $X = \{X_t, t \geq 0\}$, can be constructed. This so-called Lévy process starts at zero $X_0 = 0$, has independent and stationary increments and the distribution of the increments $X_{t+s} - X_s$, has $(\phi(u))^t$ as characteristic function.

In the present setting we work with Lévy processes only on the unit time interval. The cumulative distribution function of X_t and its inverse are denoted by H_t and $H_t^{[-1]}$ respectively. We normalise the distribution so that $\mathbb{E}[X_1] = 0$ and $\text{Var}[X_1] = 1$. Hence one has $\text{Var}[X_t] = t$. We take X and $X^{(i)}$, $i = 1, 2, \ldots, N$ independent and identically distributed Lévy processes.

[3] The [0–3%] equity tranche is quoted as an upfront payment plus 500 basis points (a basis point is equal to 0.01%) running.

[4] This idea is inspired in the original work of Merton [Mer74] known in the literature as *firm* or *asset* value models.

In the generic 1-factor Lévy model the latent variable is represented as

$$A_i = X_\rho + X_{1-\rho}^{(i)}, \quad i = 1, \dots, N \tag{9}$$

and each A_i has the same distribution function H_1. Note that for $i \neq j$, we have Corr$[A_i, A_j] = \rho$. The threshold implied from the CDS risk neutral probability of defaults $(p_i(t))$ is given by

$$K_i(t) = H_1^{[-1]}(p_i(t)).$$

Denote by $p_i(y;t)$ the conditional default probability of firm i given the value y for the systemic factor. From (9) we have

$$p_i(y;t) = H_{1-\rho}(K_i(t) - y).$$

2.4 The Shifted Gamma Model

The Gamma(a, b) distribution with parameters $a, b > 0$ is infinitely divisible. Its density function is given by

$$f(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \tag{10}$$

for $x > 0$. The Gamma-process G_t is defined as the stochastic process which starts at zero and has stationary, independent Gamma-distributed increments. The time enters in the first parameter, that is G_t follows a Gamma(at, b) distribution. We set

$$X_t = \sqrt{a}t - G_t. \tag{11}$$

It is clear that X_t is normalised in the sense that $\mathbb{E}[X_1] = 0$ and Var$[X_1] = 1$ since Gamma distributions with $b = \sqrt{a}$ have variance 1.

Both the cumulative distribution function $H_t(x; a)$ of X_t, and its inverse $H_t^{[-1]}(y; a)$, can easily be obtained from the Gamma cumulative distribution function and its inverse. Gauss-Laguerre quadrature can be used to evaluate the integral in (6).

2.5 Gaussian Copula

In the 1-factor Gaussian copula setting the latent variable A_i is given by:

$$A_i = \sqrt{\rho}X + \sqrt{1-\rho}X^{(i)}, \quad i = 1, \dots, N \tag{12}$$

the systemic factor X and the idiosyncratic factor $X^{(i)}$ are taken to be independent standard normal random variables. Gauss-Hermite quadrature can be used to evaluate the integral in (6). Note that the Gaussian copula model is a special case of the generic 1-factor Lévy model, in which the normal distribution is used.

3 Lévy Base Correlation

The shifted gamma model outlined in section 2.4 has 2 unknown parameters: the drift parameter a in (11) and the correlation parameter ρ in (9). One possible calibration scheme is to use a *global model* that calibrates the two parameters to the 5 observed tranche spreads. It is observed that sensitivity to a and ρ is quite different. Hence both a and ρ can be used to explain spread movements. This global model however is unable to match the 5 observed tranche spreads exactly.

Analogous to the Gaussian copula case, a solution is then to move to the base correlation methodology and use a base correlation value for every (base) tranche. For iTraxx we have 5 base correlation values ρ_3, ρ_6, ρ_9, ρ_{12} and ρ_{22}. We are then left with the problem of calibrating 6 parameters to the 5 observed tranche spreads. Several calibration strategies are possible. We could impose an additional constraint, such as requiring that the base correlation be equal for the equity and the junior mezzanine tranches ($\rho_3 = \rho_6$), or we could set an additional optimisation target, such as smoothness of the base correlation curve and minimise slope of the base correlation curve. All these strategies have the same annoyance: one parameter is of different nature. The a parameter is not a (base) correlation value, but rather characterises the Gamma distribution used.

To overcome this we set $a = 1$ and proceed along the base correlation methodology. Fixing $a = 1$ results in the Exponential(1) distribution being used for G_1. The motivation for this choice is as follows. First we present a distributional argument. Recall that the normal distribution has a density proportional to $\exp(-0.5x^2)$ and consequently decays very fast. The exponential distribution on the other hand with a density $f(x) = \exp(-x)$ for $x \geq 0$ is the typical representant of the next class of fatter (than normal) tail distributions. It turns out that moving into this class of distributions is a key building block. Second $a = 1$ is a typical value in free a calibration. Finally it is the most tractable choice.

In what follows Lévy base correlation is defined as the base correlation in the shifted gamma model with fixed $a = 1$. For more details on Lévy base correlation and a comparison with the Gaussian base correlation we refer to our earlier paper [GGMS07] and the references therein. Another example of a Lévy based model is given by Hooda [Hoo06]. He has shown that the Normal Gamma copula base correlation curve is less skewed than the Gaussian copula base correlation curve.

4 Historical Study

In this section we describe our historical study. The dataset used is the iTraxx Europe Main data from April 20th, 2005 until March 16th, 2007. We look at index and tranche spreads for the 5, 7 and 10 year maturity contracts. Fig. 1 shows the historical evolution of the spreads for the 5 year maturity.

The key parameters in both models are the base correlation curves. First we show the evolution over time and second we consider the behaviour across maturity and look at the base correlation surface. For trading purposes it is important to understand the hedge parameters in

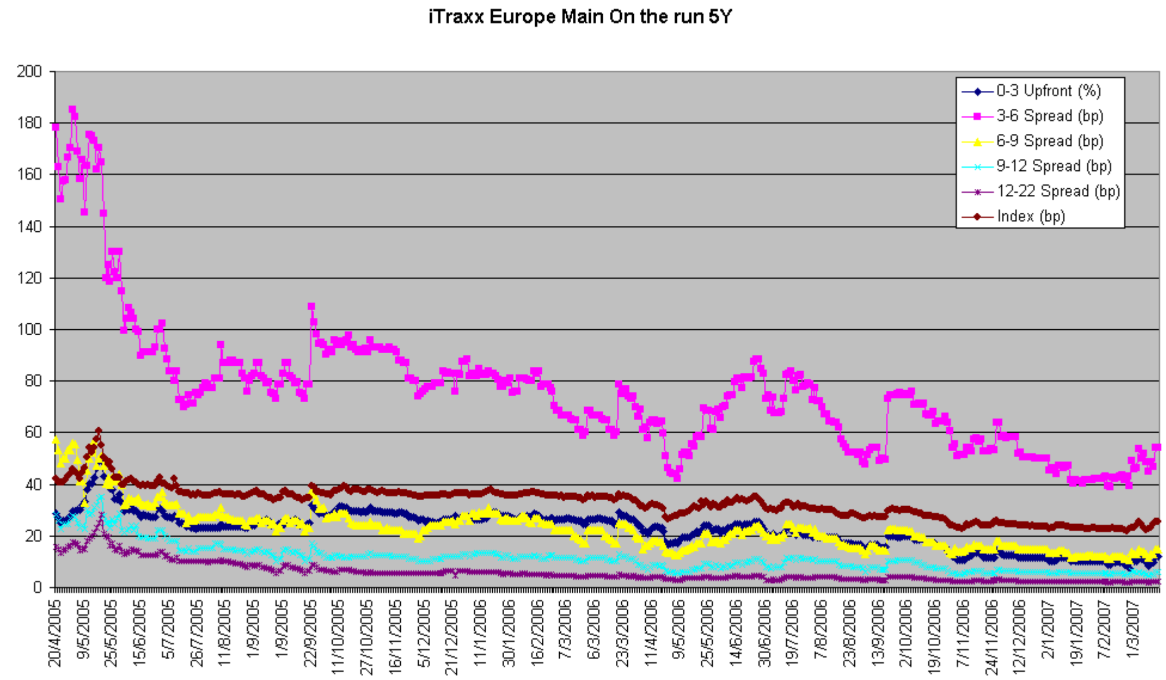

Figure 1: iTraxx Europe Main On the run 5y Spreads.

the two models. We present a comparison between the two models and focus on the deltas of the tranches with respect to the index.

5 Base Correlation

5.1 Base Correlation Evolution over Time

As mentioned above both models are parametrised with a base correlation curve. We have compared the evolution over time of the base correlation curves in both models and found that their behaviour is very similar.

The evolution over time for the 12–22 tranche for the Gaussian and Lévy base correlation models is shown in Fig. 2. We present the results for the 12–22 tranche since this one is the last one to be computed in the bootstrapping procedure and consequently it is sensitive to all previous computations. Lévy base correlation and Gaussian base correlation clearly behave in the same way, just on a different scale.

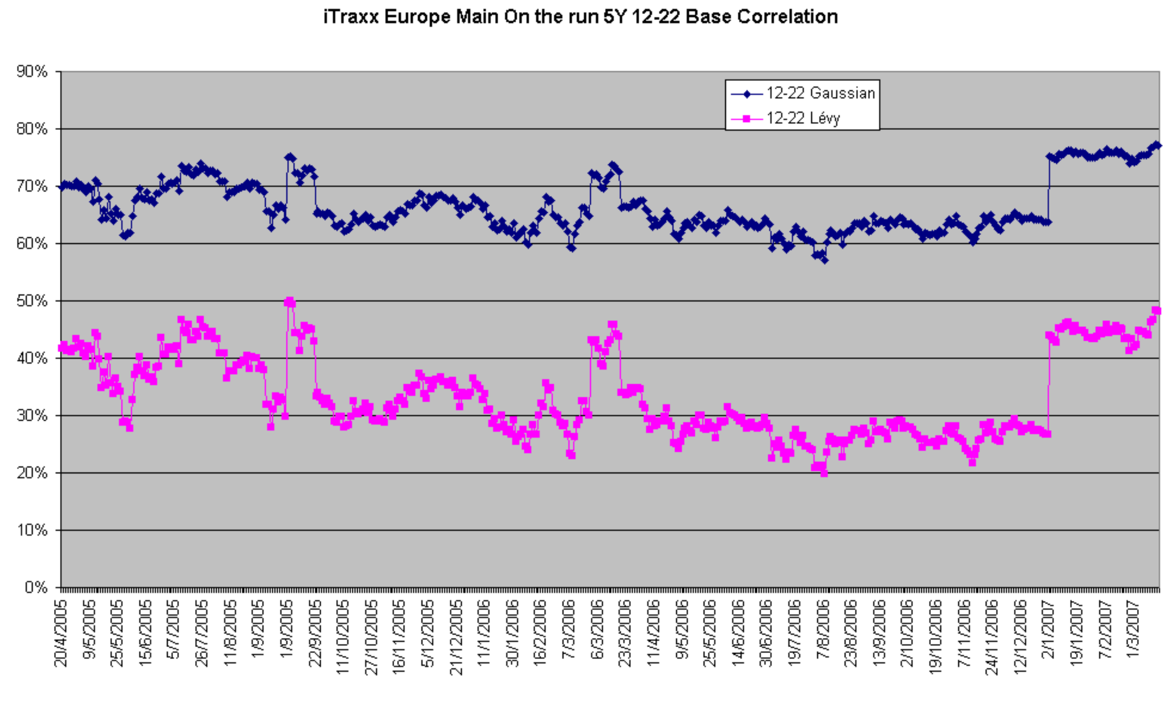

Figure 2: iTraxx Europe Main 5y Gaussian and Lévy Base Correlation for 12–22 tranche.

5.2 Base Correlation across Maturity

We now turn to the base correlation surface, that is we consider base correlation as a function of the attachment point and the maturity.

Fig. 3 shows the Gaussian copula base correlation surface. For every maturity we observe the typical upward sloping curve at the standard attachment points. The behaviour across maturity for a given attachment point is less uniform. This curve can be smiling, upward or downward sloping. Looking at these curves it should be clear that there is no widely accepted standard approach for interpolating a base correlation surface for a nonstandard attachment point or for a nonstandard maturity. Moreover interpolation schemes may not be arbitrage-free. For a detailed discussion on a more efficient way to interpolate in the base correlation framework, we refer to our earlier paper [GG07].

The Lévy base correlation surface is shown in Fig. 4. Contrary to the Gaussian case, the behaviour across maturity for a given attachment point is more uniform. There is an upward trend with maturity. We also see the typical upward sloping curve at the standard attachment points, except for the 10 year maturity, where we see a smile. In fact this is expected and the explanation is as follows. In our earlier paper [GG07], we have shown that both Gaussian and Lévy base correlation curves exhibit a smile for small attachment points. It is even more pronounced in the Lévy base correlation curve. On the other hand it is well known that for the 10 year maturity

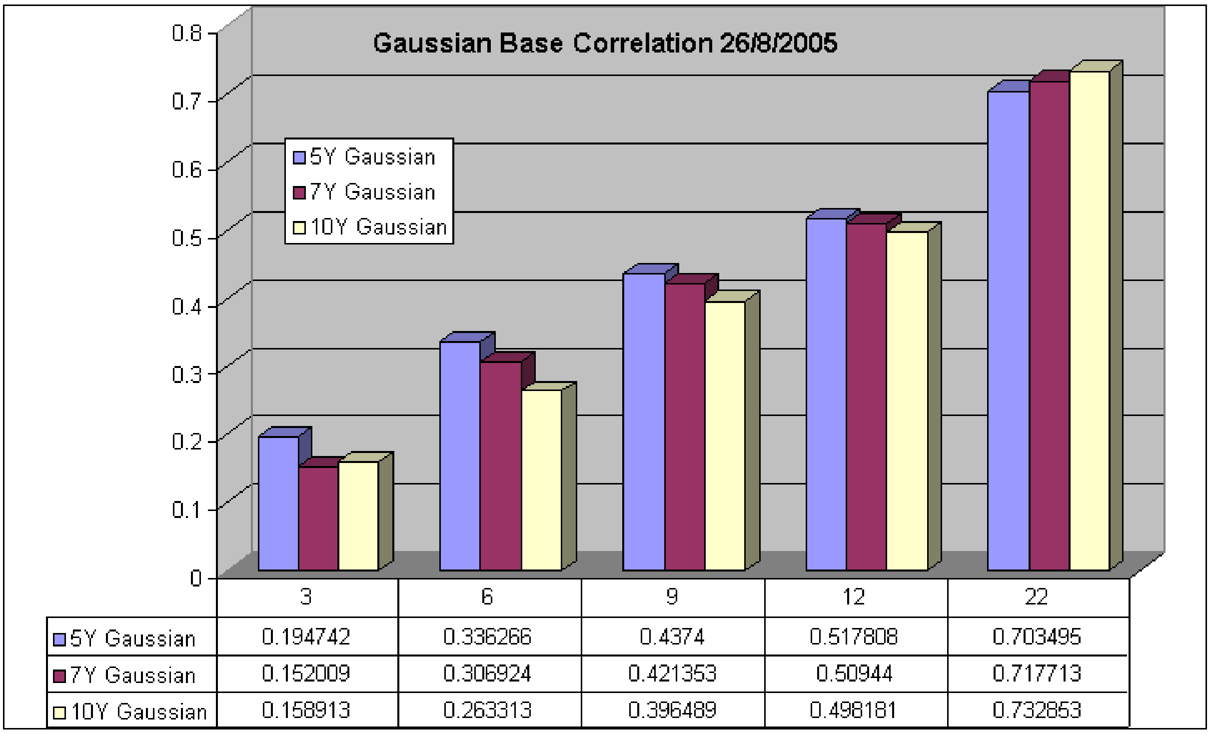

Figure 3: Gaussian Copula Base Correlation.

contract the 3–6 tranche behaves as an equity tranche. On the 10 year maturity the 0–3 equity tranche is a deeper, in the sense of lower attachment points, equity tranche. Hence we expect it to have a larger base correlation value than the 3–6 tranche.

6 Hedge Parameters

The evolution over time for the delta of the 0–3 tranche with respect to the index for the Gaussian and Lévy base correlation models is shown in Fig. 5. The values for this sensitivity parameter produced by the Gaussian and Lévy models clearly behave in the same way, just on a different scale. In order to quantify this similarity, a regression of the delta produced by the Lévy model on the delta produced by the Gaussian models is performed. This is done for all tranches and all maturities. The model is

$$\delta^{(\mathrm{L})}_{\mathrm{Tr},\,T} = \beta_{\mathrm{Tr},\,T}\,\delta^{(\mathrm{G})}_{\mathrm{Tr},\,T} \tag{13}$$

where $\delta^{(\mathrm{L})}_{\mathrm{Tr},\,T}$ is the delta of tranche Tr with respect to the underlying index, for a maturity T, produced by the Lévy model and $\delta^{(\mathrm{G})}_{\mathrm{Tr},\,T}$ is the delta produced by the Gaussian model. We now turn to the quality of these regressions.

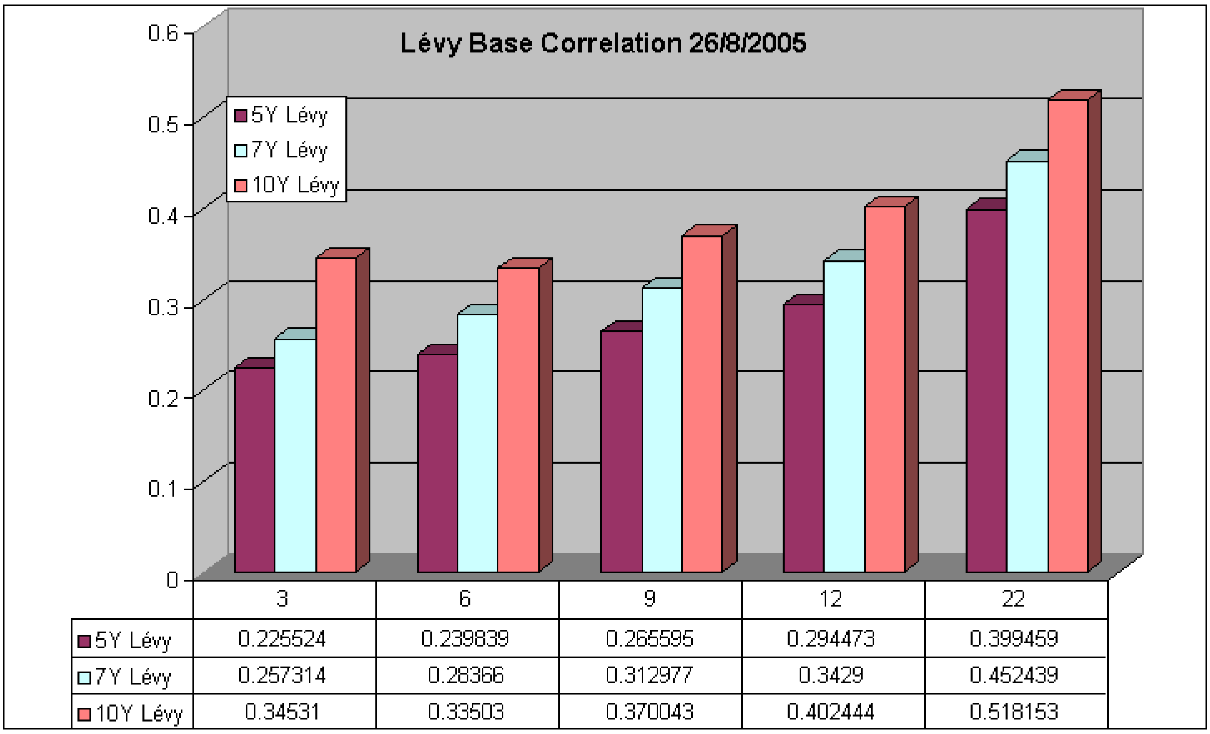

Figure 4: Lévy Base Correlation.

Table 1 shows the regression coefficient, the standard error and the coefficient of determination R^2 for each tranche for the 5 year maturity. The R^2 statistic confirms that the model (13) is a good fit. The regression results for the 7 year maturity are shown in Table 2. In this case the R^2 statistic also confirms that the model (13) is a good fit. Table 3 shows the regression results for the 10 year maturity. Also here the R^2 statistic confirms that the model (13) is a good fit.

These results show that compared to the Gaussian model, the Lévy model produces a delta that is larger for the equity tranches and smaller for the senior tranches. Roughly speaking we can say that the Lévy model delta is approximately 20% larger for the equity tranche and 40% smaller for the senior tranches. This result is to be compared to “statistical” deltas quoted by

Tr	$\beta_{\text{Tr}, T}$	Std Err	R^2
0–3	1.198891	0.001143	0.9996
3–6	0.761988	0.002592	0.9946
6–9	0.567959	0.001864	0.995
9–12	0.527024	0.004981	0.9601
12–22	0.49989	0.01036	0.8334

Table 1: Regression of Lévy delta on Gaussian delta for the 5 year maturity tranches.

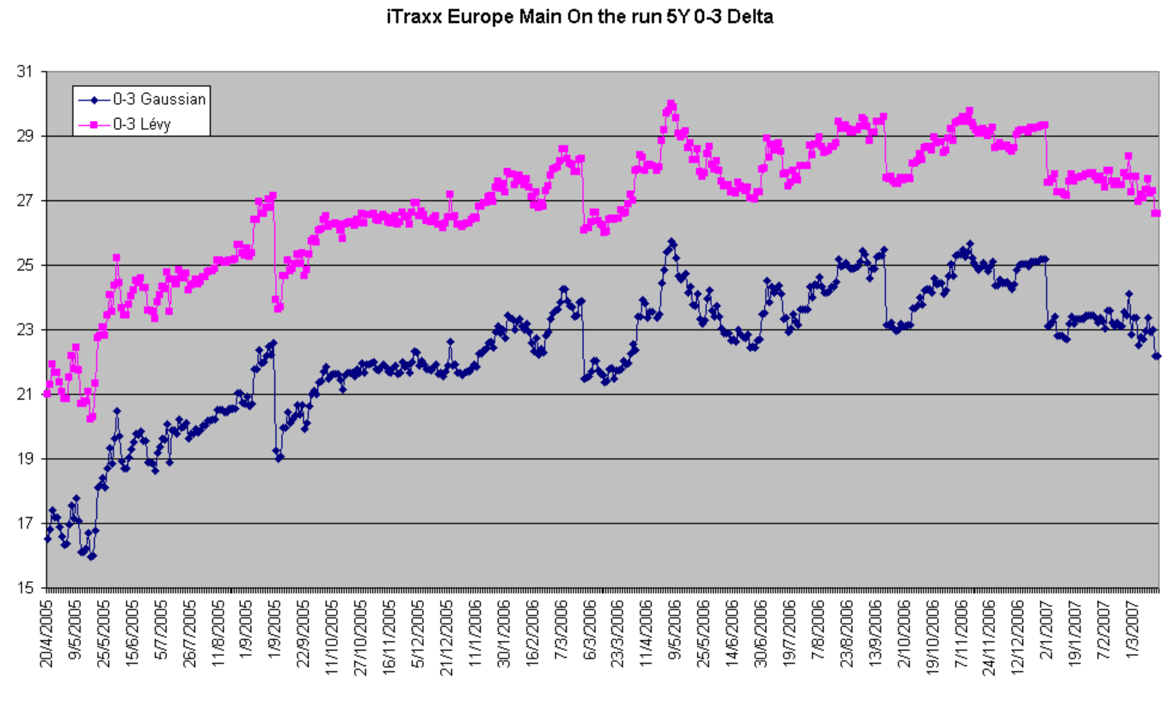

Figure 5: iTraxx Europe Main 5y Delta for 0–3 tranche Gaussian and Lévy model.

market participants which are typically 20% higher for the equity tranche.

7 Conclusions

We have compared the Gaussian copula and Lévy base correlation models. The results of a historical study of both models on the iTraxx Europe Main dataset have been presented. We have compared the evolution over time of the base correlation surfaces in both models and found that their behaviour is very similar. Hedge parameters in the different models have also been studied. We have focused on the deltas of the tranches with respect to the index and found the values for this sensitivity parameter produced by the Gaussian and Lévy models clearly behave in the same way. This is illustrated by the fact that a regression of one on the other results in a very good fit. Roughly speaking we can say that the Lévy model delta is approximately 20% larger for the equity tranche and 40% smaller for the senior tranches.

Tr	β	Std Err	R^2
0–3	1.25069	0.00104	0.9997
3–6	1.059253	0.004181	0.9928
6–9	0.638850	0.002096	0.995
9–12	0.567896	0.003036	0.9869
12–22	0.557808	0.009115	0.8893

Table 2: Regression of Lévy delta on Gaussian delta for the 7 year maturity tranches.

Tr	β	Std Err	R^2
0–3	1.182632	0.002417	0.9981
3–6	1.337550	0.002275	0.9987
6–9	0.895605	0.005092	0.9852
9–12	0.640572	0.001964	0.9956
12–22	0.622463	0.006962	0.9449

Table 3: Regression of Lévy delta on Gaussian delta for the 10 year maturity tranches.

References

[AS05] L. Andersen and J. Sidenius. CDO Pricing with Factor Models: Survey and Comments. Technical report, Bank of America, 2005.

[ASB03] L. Andersen, J. Sidenius, and S. Basu. All your hedges in one basket. *RISK*, November 2003.

[Ass06] British Bankers Association. *Credit Derivatives Report*. 2006.

[Ber96] J. Bertoin. *Lévy Processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 1996.

[DSFT03] A. Debuysscher, M. Szegö, M. Freydefront, and H. Tabe. The Fourier Transform Method. Technical report, International Strutured Finance, Moody's, 2003.

[GG07] J. Garcia and S. Goossens. Base Expected Loss explains Lévy Base Correlation Smile. Technical report, 2007. Available at `www.defaultrisk.com`.

[GGMS07] J. Garcia, S. Goossens, V. Masol, and W. Schoutens. Lévy Base Correlation. Technical Report 07-04, Section of Statistics, K.U.Leuven, 2007. Available at `www.defaultrisk.com`.

[Hoo06] S. Hooda. Explaining Base Correlation Skew Using Normal-Gamma Process. Technical report, Nomura, 2006. Available at `www.defaultrisk.com`.

[MBAW04] L. McGinty, E. Beinstein, R. Ahluwalia, and M. Watts. Introducing Base Correlations. Technical report, JPMorgan, 2004. Credit Derivatives Strategy.

[Mer74] R. Merton. On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance*, 29:449–470, 1974.

[OL04] D. O'Kane and M. Livasey. Base Correlation Explained. Technical report, Lehman Brothers, 2004. Quantative Credit Research Q3/4.

[Sat00] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2000.

[YHZ06] Y. Yang, T.R. Hurd, and X. Zhang. Saddlepoint approximation method for pricing CDO's. Technical report, McMaster University, 2006.